

TRIPLE MASSEY PRODUCTS IN GALOIS COHOMOLOGY

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ABSTRACT. Fix an arbitrary prime p . Let F be a field, containing a primitive p -th root of unity, with absolute Galois group G_F . The triple Massey product (in the mod- p Galois cohomology) is a partially defined, multi-valued function $\langle \cdot, \cdot, \cdot \rangle : H^1(G_F)^3 \rightarrow H^2(G_F)$. In this work we prove a conjecture made in [11] stating that any defined triple Massey product contains zero. As a result the pro- p groups appearing in [11] are excluded from being absolute Galois groups of fields F as above.

1. INTRODUCTION

We fix a prime number p . Let F be a field, which will always be assumed to contain a primitive p -th root of unity, ρ . Let G be a profinite group acting trivially on $\mathbb{Z}/p\mathbb{Z}$ and let $H^i(G)$ denote the i -th cohomology group $H^i(G, \mathbb{Z}/p\mathbb{Z})$. The triple Massey product is a partially defined, multi-valued function $\langle \cdot, \cdot, \cdot \rangle : H^1(G)^3 \rightarrow H^2(G)$. In particular, for $1 \leq i \leq 3$, let $\chi_i \in H^1(G)$, be such that $\chi_1 \cup \chi_2 = \chi_2 \cup \chi_3 = 0$ in $H^2(G)$. Then one can define a subset, $\langle \chi_1, \chi_2, \chi_3 \rangle \subseteq H^2(G)$, called the triple Massey product of χ_1, χ_2, χ_3 . In [11], Mináč and Tân define the Vanishing triple Massey product property of G , stating that every defined triple Massey product contains zero. They conjecture that if $G = G_F$, the absolute Galois group of F , then for any prime p , G_F has this property. When $p = 2$ they prove the conjecture and use it to show certain pro-2 groups can not be realizable as absolute Galois groups. The main objective of this work is to prove their conjecture in its full generality, i.e. for any prime p and any field F as above. This is achieved in Theorem 6.6. As a result we get a strong restriction on

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the structure of G_F , ruling out certain pro- p groups (along the lines of [11]) from being realizable as absolute Galois groups.

The Vanishing triple Massey product for G_F was known in the following cases:

- (1) In [7], for $p = 2$, Hopkins and Wickelgren construct an F -variety $X(\chi_1, \chi_2, \chi_3)$ such that $0 \in \langle \chi_1, \chi_2, \chi_3 \rangle \Leftrightarrow X_F(\chi_1, \chi_2, \chi_3) \neq \emptyset$ and prove that it always has an F -point when F is a number field.
- (2) In [11], for $p = 2$, Mináč and Tân present an F -point of $X(\chi_1, \chi_2, \chi_3)$ where F is any field. Thus proving G_F always has the Vanishing triple Massey product property with respect to the prime $p = 2$. Using the work of Dwyer (see [3]) they give examples of pro- p groups which do not possess the Vanishing triple Massey product property. Thus, for $p = 2$ they get new examples of profinite groups which can not be realized as the absolute Galois groups of the field F .
- (3) In [13] Mináč and Tân prove that if F is a number field, G_F has the Vanishing triple Massey product property with respect to any prime p .
- (4) In [4], Efrat and the author connect this property to the Brauer group of F . When $p = 2$ they show that an old Theorem of Albert (see [1]), strengthened by Rowen (see [15, Corollary 5]) implies the Vanishing triple Massey product property for G_F , where F is arbitrary. Moreover, when F is a number field they prove a result about relative Brauer groups of abelian extensions of number fields which implies the Vanishing triple Massey product property for G_F with respect to any prime p .

Following [4] we approach the problem from the point of view of the Brauer group of F . Namely, we make use of the isomorphism:

$$\Psi : H^2(G_F) \xrightarrow{\sim} \text{Br}_p(F)$$

to translate the problem to a question about specific $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ abelian crossed products and solve it using the theory of abelian crossed products. In particular, let $\chi_1, \chi_2, \chi_3 \in H^1(G_F)$, which now correspond to classes $[a_1], [a_2], [a_3] \in F^\times / (F^\times)^p$ under the Kummer isomorphism,

$$F^\times / (F^\times)^p \cong H^1(G_F).$$

Let M be any element of $\langle \chi_1, \chi_2, \chi_3 \rangle$. Then we have:

$$(1) \quad 0 \in \langle \chi_1, \chi_2, \chi_3 \rangle \Leftrightarrow \Psi(M) = [(a_1, \alpha)_p] \otimes [(a_3, \beta)_p]$$

where $\alpha, \beta \in F^\times$ and $[(a_1, \alpha)_p], [(a_3, \beta)_p]$ are the classes in $\text{Br}_p(F)$ of the symbol F -algebras $(a_1, \alpha)_p$ and $(a_3, \beta)_p$. Our strategy will be to construct an explicit abelian crossed product F -algebra A , such that $\Psi(M) = [A]$ for some element $M \in \langle \chi_1, \chi_2, \chi_3 \rangle$, and prove it satisfies the right-hand side of (1). The work is organized as follows: In section 2 we give some background on general Massey products and start the study of the specific case of triple Massey products. In section 3 we give the necessary background on abelian crossed products, construct and study a specific abelian crossed product A which will be used in the proof of the main Theorem. In section 4 we connect the two previous sections and show that every element in the triple Massey product corresponds to an abelian crossed product in the Brauer group. In Section 5 we study the condition $\chi_a \cup \chi_b = 0$ in Galois cohomology in order to better understand the corresponding abelian crossed product from section 4. In particular, we build a Galois extension with group G , and construct a concrete function $\varphi_{a,b} \in C^1(G_F)$ such that $\partial(\varphi_{a,b}) = \chi_a \cup \chi_b \in C^2(G_F)$. Section 6 is devoted to the proof of the Vanishing triple Massey product property of absolute Galois groups. In particular, we use the function from section 5 to show A (constructed in section 3) corresponds to some element in the triple Massey product and use our study of A in section 3 to prove the main Theorem.

2. n -FOLD MASSEY PRODUCTS

In this section we give the necessary background on (rank 1) n -fold Massey products $\langle \cdot, \dots, \cdot \rangle : H^1(G)^n \rightarrow H^2(G)$.

2.1. Background. The main sources for this background subsection are [3], [5], [7] and [18]. Let R be a unital commutative ring. Recall that a differential graded algebra (DGA) over R is a graded R -algebra

$$C^\bullet = \bigoplus_{k \geq 0} C^k = C^0 \oplus C^1 \oplus C^2 \oplus \dots$$

with product \cup , and equipped with a differential $\partial : C^\bullet \rightarrow C^{\bullet+1}$ such that:

- (1) $\partial(x \cup y) = \partial(x) \cup y + (-1)^k x \cup \partial(y)$ for $x \in C^k$;
- (2) $\partial^2 = 0$.

One then defines the cohomology ring $H^\bullet = \text{Ker}(\partial) / \text{Im}(\partial)$.

Definition 2.1. Let $c_1, \dots, c_n \in H^1$. A collection $C = (c_{i,j}), 1 \leq i < j \leq n+1, (i, j) \neq (1, n+1)$, of elements of C^1 is called a defining system for the n -th fold Massey product $\langle c_1, \dots, c_n \rangle$ if the following conditions hold:

- (1) $c_{i,i+1}$ represents c_i for every $1 \leq i \leq n$;
- (2) $\partial(c_{i,j}) = \sum_{k=i+1}^{j-1} c_{i,k} \cup c_{k,j}$ for every i, j as above and $i+1 < j$.

One can then check that $\sum_{k=2}^n c_{1,k} \cup c_{k,n+1}$ is in $\text{Ker}(\partial)$, thus represents an element of H^2 . It is called the n -fold Massey product with respect to the defining system C , denoted $\langle c_1, \dots, c_n \rangle_C$. Then

$$\langle c_1, \dots, c_n \rangle = \{ \langle c_1, \dots, c_n \rangle_C \mid C \text{ is a defining system} \}.$$

Remark 2.2. For $n = 2$, $\langle c_1, c_2 \rangle = \{c_1 \cup c_2\}$.

2.2. Massey products and unipotent representations. Let G be a profinite group and let R be a finite commutative ring considered as a trivial discrete G -module. Let $C^\bullet = C^\bullet(G, R)$ be the DGA of inhomogeneous continuous cochains of G with coefficients in R [14, Chapter I Section 2]. In [3], Dwyer shows in the discrete context (see also [5, section 8] in the profinite case), that defining systems for this DGA can be interpreted in terms of upper-triangular unipotent representations of G , in the following way.

Let $\mathbb{U}_{n+1}(R)$ be the group of all upper-triangular unipotent $(n+1) \times (n+1)$ -matrices over R . Let $Z_{n+1}(R)$ be the subgroup of all such matrices with entries only at the $(1, n+1)$ position. We may identify $\mathbb{U}_{n+1}(R)/Z_{n+1}(R)$ with the group $\bar{\mathbb{U}}_{n+1}(R)$ of all upper-triangular unipotent $(n+1) \times (n+1)$ -matrices over R where we remove the $(1, n+1)$ -entry.

For a representation $\varphi : G \rightarrow \mathbb{U}_{n+1}(R)$ and $1 \leq i < j \leq n+1$ let $\varphi_{i,j} : G \rightarrow R$ be the composition of φ with the projection from $\mathbb{U}_{n+1}(R)$ to its (i, j) -coordinate; and the same for a representation $\bar{\varphi} : G \rightarrow \bar{\mathbb{U}}_{n+1}(R)$.

Theorem 2.3. ([3, Theorem 2.4]) Let c_1, \dots, c_n be elements of $H^1(G, R)$. There is a one-one correspondence $C \leftrightarrow \bar{\varphi}_C$ between defining systems C for $\langle c_1, \dots, c_n \rangle$ and group homomorphisms $\bar{\varphi}_C : G \rightarrow \bar{\mathbb{U}}_{n+1}(R)$ with $\bar{\varphi}_{i,i+1} = c_i$, for $1 \leq i \leq n$. Moreover, $\langle c_1, \dots, c_n \rangle_C = 0$ if and only if the

dotted arrow exists in the following commutative diagram

$$\begin{array}{ccccccc}
 & & & & G & & \\
 & & & \nearrow \text{dotted} & \downarrow \bar{\varphi}_C & & \\
 1 & \longrightarrow & R & \longrightarrow & \mathbb{U}_{n+1}(R) & \longrightarrow & \bar{\mathbb{U}}_{n+1}(R) \longrightarrow 1.
 \end{array}$$

Definition 2.4. Following [11], for $n \geq 2$ we say that G has the Vanishing n -fold Massey product property if every defined Massey product $\langle c_1, \dots, c_n \rangle$, contains zero.

2.3. Triple Massey products in Galois cohomology. Let F be a field containing a primitive p -th root of unity, ρ , with absolute Galois group G_F . We now focus on the case $n = 3$ where the DGA used is that of Galois cohomology, namely $C^\bullet = C^\bullet(G_F, \mathbb{Z}/p\mathbb{Z})$, abbreviated $C^\bullet(G_F)$. In this case the above construction can be described as follows: Let $\chi_a, \chi_b, \chi_c \in H^1(G_F)$ such that

$$(2) \quad \chi_a \cup \chi_b = \chi_b \cup \chi_c = 0 \in H^2(G_F).$$

Then there exist $\varphi = \{\varphi_{a,b}, \varphi_{b,c}\} \subset C^1(G_F)$ such that

- (1) $\partial(\varphi_{a,b}) = \chi_a \cup \chi_b$ in $C^2(G_F)$.
- (2) $\partial(\varphi_{b,c}) = \chi_b \cup \chi_c$ in $C^2(G_F)$.

We call such φ a defining system. Then

$$\langle \chi_a, \chi_b, \chi_c \rangle_\varphi = \chi_a \cup \varphi_{b,c} + \varphi_{a,b} \cup \chi_c.$$

Note that

$$\partial(\langle \chi_a, \chi_b, \chi_c \rangle_\varphi) = \partial(\chi_a \cup \varphi_{b,c} + \varphi_{a,b} \cup \chi_c) = \chi_a \cup \chi_b \cup \chi_c - \chi_a \cup \chi_b \cup \chi_c = 0.$$

Hence $\langle \chi_a, \chi_b, \chi_c \rangle_\varphi$ represents a class in $H^2(G_F)$ which we also denote by $\langle \chi_a, \chi_b, \chi_c \rangle_\varphi$.

Then assuming condition (2) holds, the triple Massey product is:

$$\langle \chi_a, \chi_b, \chi_c \rangle = \{ \langle \chi_a, \chi_b, \chi_c \rangle_\varphi \mid \varphi \text{ is a defining system} \} \subseteq H^2(G_F).$$

Remark 2.5. Notice that once a defining system $\varphi = \{\varphi_{a,b}, \varphi_{b,c}\}$ is chosen, $\varphi_{a,b}, \varphi_{b,c}$ are unique up to elements of $H^1(G_F)$, thus the set of all possible $\langle \chi_a, \chi_b, \chi_c \rangle_\varphi$ is a coset

$$\langle \chi_a, \chi_b, \chi_c \rangle = \langle \chi_a, \chi_b, \chi_c \rangle_\varphi + \chi_a \cup H^1(G_F) + H^1(G_F) \cup \chi_c.$$

Proof. Indeed by definition of cohomology, for $\chi \in H^1(G_F)$ one has $\partial(\chi) = 0$, and vice versa. \square

This clearly implies,

Proposition 2.6. *Let φ be any defining system for $\langle \chi_a, \chi_b, \chi_c \rangle$. Then,*

$$0 \in \langle \chi_a, \chi_b, \chi_c \rangle \Leftrightarrow \langle \chi_a, \chi_b, \chi_c \rangle_\varphi \in \chi_a \cup H^1(G_F) + H^1(G_F) \cup \chi_c.$$

3. ABELIAN CROSSED PRODUCTS

In this section we give the necessary background on abelian crossed products, define and study a specific abelian crossed product which we will later use.

3.1. Background. A crossed product is a central simple algebra with a maximal subfield Galois over the center. Such algebras, with an abelian Galois group, were studied by Amitsur and Saltman in [2]. Our focus is in degree p^2 , so let us summarize what we need from their work.

Theorem 3.1. *Let K/F be a Galois extension of fields with Galois group*

$$\text{Gal}(K/F) = \langle \sigma_1, \sigma_2 \rangle \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z},$$

and let $b_1, b_2, u \in K^\times$ be elements satisfying the equations

$$\begin{aligned} \sigma_i(b_i) &= b_i, \\ \sigma_2(b_1)b_1^{-1} &= N_{K/K^{\sigma_1}}(u), \\ \sigma_1(b_2)b_2^{-1} &= N_{K/K^{\sigma_2}}(u)^{-1}. \end{aligned} \tag{3}$$

Then the algebra $A = K[z_1, z_2]$, defined by the relations $z_i k z_i^{-1} = \sigma_i(k)$ for $k \in K$, $z_i^p = b_i$ and $z_2 z_1 = u z_1 z_2$, is an F -central simple algebra containing K as a maximal subfield, and every such algebra has this form.

The crossed product defined above is denoted $(K/F, \{\sigma_1, \sigma_2\}, \{b_1, b_2, u\})$.

3.2. Construction of an abelian crossed product A_{v_1, v_2} .

Let $a_1, a_2 \in F^\times$ and let $F_i = F[x_i | x_i^p = a_i]$, $i = 1, 2$, be the corresponding Kummer extensions of F , with Galois groups, $G_i = \text{G}(F_i/F) = \langle \sigma_i \rangle \cong \mathbb{Z}/p\mathbb{Z}$. Let $K = F_1 F_2$. Then $\text{G}(K/F) = \langle \sigma_1, \sigma_2 \rangle$. Note that $K^{\sigma_1} = F_2$, $K^{\sigma_2} = F_1$. Assume that there exist elements $v_i \in F_i$, $i = 1, 2$, such that

$$(4) \quad N_{F_{a_1}/F}(v_1) = N_{F_{a_2}/F}(v_2).$$

Define

$$u = \frac{v_2}{v_1}.$$

Notice that $N_{K/F}(u) = 1$.

Proposition 3.2. *Let*

$$w_1 = \prod_{i=0}^{p-1} \sigma_2^i(v_2)^i \in F_2, \quad w_2 = \prod_{i=0}^{p-1} \sigma_1^i(v_1)^i \in F_1.$$

Then we have:

$$\begin{aligned} (1) \quad N_{K/F_2}(u) &= \frac{\sigma_2(w_1)}{w_1}. \\ (2) \quad N_{K/F_1}(u)^{-1} &= \frac{\sigma_1(w_2)}{w_2}. \end{aligned}$$

Proof. (1) We compute

$$\begin{aligned} \sigma_2(w_1) &= \sigma_2\left(\prod_{i=0}^{p-1} \sigma_2^i(v_2)^i\right) \\ &= \sigma_2^2(v_2) \sigma_2^3(v_2)^2 \cdots \sigma_2^{p-1}(v_2)^{p-2} v_2^{p-1} \\ &= \frac{\prod_{i=0}^{p-1} \sigma_2^i(v_2)}{\prod_{i=0}^{p-1} \sigma_2^i(v_2)} \sigma_2^2(v_2) \sigma_2^3(v_2)^2 \cdots \sigma_2^{p-1}(v_2)^{p-2} v_2^{p-1} \\ &= \frac{v_2^p \prod_{i=0}^{p-1} \sigma_2^i(v_2)^i}{N_{F_2/F}(v_2)} = \frac{v_2^p}{N_{F_1/F}(v_1)} w_1 = N_{K/F_2}\left(\frac{v_2}{v_1}\right) w_1 = N_{K/F_2}(u) w_1. \end{aligned}$$

Thus, $N_{K/F_2}(u) = \frac{\sigma_2(w_1)}{w_1}$.

(2) We compute

$$\begin{aligned} \sigma_1(w_2) &= \sigma_1\left(\prod_{i=0}^{p-1} \sigma_1^i(v_1)^i\right) \\ &= \sigma_1^2(v_1) \sigma_1^3(v_1)^2 \cdots \sigma_1^{p-1}(v_1)^{p-2} v_1^{p-1} \\ &= \frac{\prod_{i=0}^{p-1} \sigma_1^i(v_1)}{\prod_{i=0}^{p-1} \sigma_1^i(v_1)} \sigma_1^2(v_1) \sigma_1^3(v_1)^2 \cdots \sigma_1^{p-1}(v_1)^{p-2} v_1^{p-1} \\ &= \frac{v_1^p \prod_{i=0}^{p-1} \sigma_1^i(v_1)^i}{N_{F_1/F}(v_1)} = \frac{v_1^p}{N_{F_2/F}(v_2)} w_2 = N_{K/F_1}\left(\frac{v_1}{v_2}\right) w_2 = N_{K/F_1}(u)^{-1} w_2. \end{aligned}$$

Thus, $N_{K/F_1}(u)^{-1} = \frac{\sigma_1(w_2)}{w_2}$. □

We have proved:

Proposition 3.3. *Let $v_1, v_2, w_1, w_2 \in K^\times$ be as in (4), (5) respectively. Consider the following data:*

$$u = \frac{v_2}{v_1}, \quad b_1 = w_2, \quad b_2 = w_1.$$

Then all the conditions of (3) are satisfied and there exist an abelian crossed product,

$$A_{v_1, v_2} = (K/F, \{\sigma_1, \sigma_2\}, \{w_2, w_1, u\}).$$

We recall that this means that $A_{v_1, v_2} = K[z_1, z_2]$ such that the following relations hold,

$$\begin{aligned} z_i k z_i^{-1} &= \sigma_i(k) \text{ for } i = 1, 2 \text{ and all } k \in K, \\ z_1^p &= w_2, \quad z_2^p = w_1, \\ z_2 z_1 &= u z_1 z_2. \end{aligned}$$

For the rest of this subsection we will show that

$$A_{v_1, v_2} \cong (a, \alpha)_p \otimes (c, \beta)_p$$

for some $\alpha, \beta \in F^\times$.

Lemma 3.4. (1) *Let $g = \sigma_1 \sigma_2$, then $N_{K/K\langle g \rangle}(u) = 1$.*

(2) *There exist $t \in K$ such that $u = \frac{g(t)}{t}$.*

Proof.

(1) We compute:

$$\begin{aligned} N_{K/K\langle g \rangle}(u) &= \frac{N_{K/K\langle g \rangle}(v_2)}{N_{K/K\langle g \rangle}(v_1)} = \frac{\prod_{i=0}^{p-1} g^i(v_2)}{\prod_{i=0}^{p-1} g^i(v_1)} \\ &= \frac{\prod_{i=0}^{p-1} (\sigma_1 \sigma_2)^i(v_2)}{\prod_{i=0}^{p-1} (\sigma_1 \sigma_2)^i(v_1)} = \frac{\prod_{i=0}^{p-1} (\sigma_2)^i(v_2)}{\prod_{i=0}^{p-1} (\sigma_1)^i(v_1)} \\ &= \frac{N_{K/F_1}(v_2)}{N_{K/F_2}(v_1)} = 1. \end{aligned}$$

(2) This follows from (1) and Hilbert 90. □

We also need some computational results which we now prove.

Proposition 3.5. *Define the following elements of A_{v_1, v_2} :*

$$Z = z_1 z_2, \quad W = t z_2, \quad X = x_1^{-1} x_2, \quad Y = x_1.$$

Then we have:

$$(1) \quad X^p = \frac{a_2}{a_1}.$$

- (2) $W^p \in F$.
- (3) $WX = \rho XW$.
- (4) $Y^p = a_1$.
- (5) $Z^p \in F$.
- (6) $ZY = \rho YZ$.
- (7) $ZX = XZ$.
- (8) $ZW = WZ$.
- (9) $YX = XY$.
- (10) $YW = WY$.

Proof.

(1), (4) Recall that $K = F[x_1, x_2]$ and $x_1^p = a_1$, $x_2^p = a_2$. Now we compute: $X^p = \left(\frac{x_2}{x_1}\right)^p = \frac{a_2}{a_1}$. Statement (4) follows by a similar computation.

(7), (8), (9), (10) We prove statement (8) and note that statements (7), (9) and (10) follow by similar computations. We compute:

$$\begin{aligned}
 ZWZ^{-1}W^{-1} &= z_1z_2tz_2(z_1z_2)^{-1}(tz_2)^{-1} \\
 &= z_1z_2tz_2z_2^{-1}z_1^{-1}z_2^{-1}t^{-1} \\
 &= z_1z_2tz_1^{-1}z_2^{-1}t^{-1} \\
 &= z_1z_2z_1^{-1}z_2^{-1}\sigma_2\sigma_1(t)t^{-1} = u^{-1}u = 1.
 \end{aligned}$$

(2), (5) First notice that W acts by conjugation on K as σ_2 , indeed for $k \in K$ we have

$$WkW^{-1} = tz_2k(tz_2)^{-1} = tz_2kz_2^{-1}t^{-1} = \sigma_2(k).$$

As σ_2 is of order p , we see that W^p acts by conjugation on K as the identity, so W^p commutes with K . But K is a maximal subfield, thus $C_{A_{v_1, v_2}}(K) = K$, implying that $W^p \in K$. Now $\text{Gal}(K/F) = \langle g, \sigma_2 \rangle = \langle \sigma_1\sigma_2, \sigma_2 \rangle$, and it is enough to show W^p is invariant under its action. Notice that Z acts on K by conjugation as $g = \sigma_1\sigma_2$. Hence,

$$g(W^p) = ZW^pZ^{-1} \stackrel{(8)}{=} W^p$$

and

$$\sigma_2(W^p) = WW^pW^{-1} = W^p$$

and statement (2) follows.

Statement (5) follows by a similar computation.

(3), (6) We compute:

$$\begin{aligned} WX &= tz_2x_1^{-1}x_2 = x_1^{-1}tz_2x_2 = x_1^{-1}t\rho x_2z_2 \\ &= \rho x_1^{-1}x_2tz_2 = \rho XW. \end{aligned}$$

Statement (6) follows by a similar computation. \square

Finally we have:

Theorem 3.6.

$$A_{v_1, v_2} \cong (a_1, W^{-p})_p \otimes (a_2, W^p Z^p)_p.$$

Proof. Define two subalgebras of A_{v_1, v_2} :

$$A_1 = F[X, W] \quad , \quad A_2 = F[Y, Z].$$

First note that by (1)–(3) of Proposition 3.5 we see that $A_1 = (\frac{a_2}{a_1}, W^p)_p$ and by (4) – (6) of the same Proposition we see that $A_2 = (a_2, Z^p)_p$. Now from (7) – (10) of Proposition 3.5 we get that A_1, A_2 commute element-wise. Thus, by dimension count and the Double Centralizer Theorem (see [16, Theorem 24.32]) we get that

$$A_{v_1, v_2} \cong A_1 \otimes A_2 = (a_1^{-1}a_2, W^p)_p \otimes (a_2, Z^p)_p$$

and the Theorem follows. \square

4. TRIPLE MASSEY PRODUCTS AND ABELIAN CROSSED PRODUCTS

In this section we connect triple Massey products and abelian crossed products.

Recall from subsection 2.3 that given $\chi_a, \chi_b, \chi_c \in H^1(G_F)$ such that

$$\chi_a \cup \chi_b = \chi_b \cup \chi_c = 0,$$

a defining system is a subset $\varphi = \{\varphi_{a,b}, \varphi_{a,b}\} \subset C^1(G_F)$ such that

$$\partial(\varphi_{a,b}) = \chi_a \cup \chi_b \text{ and } \partial(\varphi_{b,c}) = \chi_b \cup \chi_c$$

in $C^2(G_F)$. For every defining system one constructs

$$\langle \chi_a, \chi_b, \chi_c \rangle_\varphi = \chi_a \cup \varphi_{b,c} + \varphi_{a,b} \cup \chi_c$$

and the triple Massey product of χ_a, χ_b, χ_c is

$$\langle \chi_a, \chi_b, \chi_c \rangle = \{ \langle \chi_a, \chi_b, \chi_c \rangle_\varphi \mid \varphi \text{ is a defining system} \} \subseteq H^2(G_F).$$

Proposition 4.1. *For a defining system φ as above we have:*

$$(1) \text{ res}_{\text{Ker}(\chi_a)}(\langle \chi_a, \chi_b, \chi_c \rangle_\varphi) = \varphi_{a,b} \cup \chi_c \in H^2(\text{Ker}(\chi_a)).$$

- (2) $\text{res}_{\text{Ker}(\chi_c)}(\langle \chi_a, \chi_b, \chi_c \rangle_\varphi) = \chi_a \cup \varphi_{b,c} \in H^2(\text{Ker}(\chi_c)).$
- (3) $\text{res}_{\text{Ker}(\chi_a) \cap \text{Ker}(\chi_c)}(\langle \chi_a, \chi_b, \chi_c \rangle_\varphi) = 0.$

Proof. (1) Let $g_1, g_2 \in \text{Ker}(\chi_a)$ one has:

$$\begin{aligned}
 \langle \chi_a, \chi_b, \chi_c \rangle_\varphi(g_1, g_2) &= (\chi_a \cup \varphi_{b,c} + \varphi_{a,b} \cup \chi_c)(g_1, g_2) \\
 &= \chi_a \cup \varphi_{b,c}(g_1, g_2) + \varphi_{a,b} \cup \chi_c(g_1, g_2) \\
 &= \chi_a(g_1) \cdot \varphi_{b,c}(g_2) + \varphi_{a,b}(g_1) \cdot \chi_c(g_2) \\
 &= 0 \cdot \varphi_{b,c}(g_2) + \varphi_{a,b}(g_1) \cdot \chi_c(g_2) = \varphi_{a,b} \cup \chi_c(g_1, g_2).
 \end{aligned}$$

Thus, $\text{res}_{\text{Ker}(\chi_a)}(\langle \chi_a, \chi_b, \chi_c \rangle_\varphi) = \varphi_{a,b} \cup \chi_c \in H^2(\text{Ker}(\chi_a)).$

Statement (2) follows from a similar computation.

(3) Let $g_1, g_2 \in \text{Ker}(\chi_a) \cap \text{Ker}(\chi_c)$ one has:

$$\begin{aligned}
 \langle \chi_a, \chi_b, \chi_c \rangle_\varphi(g_1, g_2) &= (\chi_a \cup \varphi_{b,c} + \varphi_{a,b} \cup \chi_c)(g_1, g_2) \\
 &= \chi_a \cup \varphi_{b,c}(g_1, g_2) + \varphi_{a,b} \cup \chi_c(g_1, g_2) \\
 &= \chi_a(g_1) \cdot \varphi_{b,c}(g_2) + \varphi_{a,b}(g_1) \cdot \chi_c(g_2) \\
 &= 0 \cdot \varphi_{b,c}(g_2) + \varphi_{a,b}(g_1) \cdot 0 = 0.
 \end{aligned}$$

Thus, $\text{res}_{\text{Ker}(\chi_a) \cap \text{Ker}(\chi_c)}(\langle \chi_a, \chi_b, \chi_c \rangle_\varphi) = 0$ and we are done. \square

Proposition 4.2. *Every $\langle \chi_a, \chi_b, \chi_c \rangle_\varphi \in \langle \chi_a, \chi_b, \chi_c \rangle$ corresponds to a class $[A] \in \text{Br}(F)$, which is represented by a $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ abelian crossed product, A .*

Proof. Let $[A] \in \text{Br}(F)$ correspond to $\langle \chi_a, \chi_b, \chi_c \rangle_\varphi$ under the isomorphism (1). By Remark 4.1 we have $\text{res}_{\text{Ker}(\chi_a) \cap \text{Ker}(\chi_c)}(\langle \chi_a, \chi_b, \chi_c \rangle_\varphi) = 0$. Thus $[A]$ is split by the field extension corresponding to $\text{Ker}(\chi_a) \cap \text{Ker}(\chi_c)$, which is $F[a^{\frac{1}{p}}, c^{\frac{1}{p}}]$ with Galois group isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Hence by [16, Corollary 24.37], $[A]$ has a representative, a F -central simple algebra A , which is a $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ abelian crossed product. \square

Remark 4.3. *In [4], the authors show that Proposition 4.2, together with an old Theorem Albert (see [1]), strengthened by Rowen ([15, Corollary 5]), has an immediate corollary stating that when $p = 2$, G_F has the Vanishing triple Massey product property.*

5. COMPUTATION OF $\varphi_{a,b}$

Knowing Proposition 4.2, was enough for the $p = 2$ case. However, when $p \geq 3$ there is no generalization of Albert's Theorem. In fact,

works of Tignol and McKinnie (see [17], [10] respectively), show that for $p \geq 3$ there exist indecomposable abelian crossed products of exponent p and degree p^2 , thus can not be similar to the tensor product of two degree p symbol algebras. Hence, in order to solve the general case we need to get more information on the triple Massey product. In particular, we will need to better understand $\varphi_{a,b}, \varphi_{b,c}$ in the definition of the triple Massey product. To this end we assume

$$\chi_a \cup \chi_b = 0,$$

and find explicit $\varphi_{a,b} \in C^1(G_F)$ such that

$$\partial(\varphi_{a,b}) = \chi_a \cup \chi_b.$$

Let $a, b \in F^\times$ be such that their corresponding Kummer characters, χ_a, χ_b are linearly independent in $H^2(G_F)$. Assume that $\chi_a \cup \chi_b = 0$ in $H^2(G_F)$. Let

$$F_a = F[x_a | x_a^p = a] \quad , \quad F_b = F[x_b | x_b^p = b]$$

be the Kummer extensions with Galois groups

$$G_a = \langle \sigma_a \rangle \cong \mathbb{Z}/p\mathbb{Z} \quad , \quad G_b = \langle \sigma_b \rangle \cong \mathbb{Z}/p\mathbb{Z};$$

where we have $\sigma_a(x_a) = \rho x_a$ and $\sigma_b(x_b) = \rho x_b$.

Let $L = F_a F_b$ with Galois group

$$G_{ab} = \langle \sigma_a, \sigma_b \rangle \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}.$$

By Wedderburn [16, Remark 24.46] we know there exists $v \in F_b$ such that $N_{F_b/F}(v) = a$. For $i = 0, 1, \dots, p-1$ let $v_i = \sigma_b^i(v)$ and define

$$w = \prod_{i=0}^{p-1} v_i^i \in F_b.$$

Note that by Proposition 3.2 we have:

$$\sigma_b(w) = \frac{v_0^p}{a} w \quad \text{and} \quad \sigma_a(w) = w.$$

We want to construct a Galois extension containing L and $w^{\frac{1}{p}}$.

Proposition 5.1. *w is not a p -th power in L .*

Proof. Assume on the contrary that $w = t^p$ for some $t \in L$. We compute:

$$\sigma_b(t)^p = \sigma_b(w) = \frac{v_0^p}{a} w = \left(\frac{v_0}{x_a} \right)^p w = \left(\frac{v_0}{x_a} \right)^p t^p.$$

Thus we get

$$\left(\frac{\sigma_b(t)}{t}\right)^p = \left(\frac{v_0}{x_a}\right)^p.$$

Now by our assumption L is a field, hence

$$\frac{\sigma_b(t)}{t} = \rho^i \frac{v_0}{x_a}$$

for some i . In particular

$$(5) \quad \frac{\sigma_b(t)}{t} \in x_a^{-1} F_b.$$

On the other hand we have:

$$\sigma_a(t)^p = \sigma_a(w) = w = t^p.$$

Thus we get: $\left(\frac{\sigma_a(t)}{t}\right)^p = 1$. Again by our assumption L is a field, hence

$$\frac{\sigma_a(t)}{t} = \rho^j$$

for some j . Hence, $t = x_a^j u$ for $u \in L^{\sigma_a} = F_b$ as an eigenvector of σ_a with eigenvalue ρ^j . Now we compute:

$$\frac{\sigma_b(t)}{t} = \frac{\sigma_b(x_a^j u)}{x_a^j u} = \frac{\sigma_b(u)}{u}.$$

Thus

$$(6) \quad \frac{\sigma_b(t)}{t} \in F_b.$$

But as $x_a \notin F_b$ we get a contradiction with (1) and we are done. \square

Corollary 5.2. $K = L[x_w | x_w^p = w]$ is a field extension of F of dimension p^3 .

Theorem 5.3. K/F is Galois.

Proof. To prove the theorem we need to produce p^3 distinct F -automorphisms of K . Clearly we have the automorphism τ defined by

$$\tau(x_w) = \rho x_w, \quad \tau(x_a) = x_a, \quad \tau(x_b) = x_b.$$

Also as $w \in F_b$ we have $K \cong F_b[x_w] \otimes F_a$. Thus we can extend σ_a to K by setting $\sigma_a(x_w) = x_w$.

The hard part is to extend σ_b from L to K . However as we know

$$\sigma_b(w) = \frac{v_0^p}{a} w = \left(\frac{v_0}{x_a}\right)^p w,$$

we get that the minimal polynomial of $\sigma_b(x_w)$ over L in a Galois closure is the conjugate by σ_b of the minimal polynomial of x_w , namely,

$$\sigma_b(X^p - w) = X^p - \left(\frac{v_0}{x_a}\right)^p w.$$

Thus, all we have to do is send x_w to a root of $X^p - \left(\frac{v_0}{x_a}\right)^p w$. Now as the roots of $X^p - \left(\frac{v_0}{x_a}\right)^p w$ are:

$$\{\rho^i \frac{v_0}{x_a} x_w \mid i = 0, 1, \dots, p-1\};$$

all we need to do is choose one of them. We choose to extend σ_b by setting

$$\sigma_b(x_w) = \frac{v_0}{x_a} x_w.$$

Thus we have produced all the necessary automorphisms and we are done. \square

Proposition 5.4. $\text{Gal}(K/F) = \langle \sigma_a, \sigma_b, \tau \rangle$ which we denote by G , has the following relations:

- (1) $\sigma_b \sigma_a = \tau \sigma_a \sigma_b$.
- (2) The center, $\text{Cent}(G)$, of G is $\langle \tau \rangle$.

Proof. (1) It is enough to check (1) on the generators of K over F . We have:

- (i) $\sigma_b \sigma_a(x_b) = \sigma_b(x_b) = \rho x_b = \tau(\rho x_b) = \tau \sigma_a(\rho x_b) = \tau \sigma_a \sigma_b(x_b)$.
- (ii) $\sigma_b \sigma_a(x_a) = \sigma_b(\rho x_a) = \rho x_a = \tau(\rho x_a) = \tau \sigma_a(x_a) = \tau \sigma_a \sigma_b(x_a)$.
- (iii) $\sigma_b \sigma_a(x_w) = \sigma_b(x_w) = \frac{v_0}{x_a} x_w = \tau(\rho^{-1} \frac{v_0}{x_a} x_w) = \tau \sigma_a(\frac{v_0}{x_a} x_w) = \tau \sigma_a \sigma_b(x_w)$.

Thus we have proved that $\sigma_b \sigma_a = \tau \sigma_a \sigma_b$.

(2) First we show $\langle \tau \rangle \subseteq \text{Cent}(G)$:

Again, we check on the generators of K over F .

We first show: $\sigma_a \tau = \tau \sigma_a$.

- (i) $\sigma_a \tau(x_a) = \sigma_a(x_a) = \rho x_a = \tau(\rho x_a) = \tau \sigma_a(x_a)$.
The same computation shows that $\sigma_a \tau(x_b) = \tau \sigma_a(x_b)$.
- (ii) $\sigma_a \tau(x_w) = \sigma_a(\rho x_w) = \rho x_w = \tau(x_w) = \tau \sigma_a(x_w)$.

Thus we have proved that $\sigma_a \tau = \tau \sigma_a$.

As for $\sigma_b \tau = \tau \sigma_b$ we compute:

- (i) The exact same computation as above shows that $\sigma_b \tau(x_a) = \tau \sigma_b(x_a)$ and $\sigma_b \tau(x_b) = \tau \sigma_b(x_b)$.

$$(ii) \sigma_b \tau(x_w) = \sigma_b(\rho x_w) = \rho \frac{v_0}{x_a} x_w = \tau(\frac{v_0}{x_a} x_w) = \tau \sigma_b(x_w).$$

Thus we proved that $\sigma_b \tau = \tau \sigma_b$, that is $\langle \tau \rangle \subseteq \text{Cent}(G)$. We now move to proving $\text{Cent}(G) \subseteq \langle \tau \rangle$. To this end let $g = \sigma_b^i \sigma_a^j \tau^k \in \text{Cent}(G)$, then we have, $\sigma_b g = g \sigma_b$. But on the one hand

$$\sigma_b g = \sigma_b^{i+1} \sigma_a^j \tau^k.$$

On the other hand we have:

$$g \sigma_b = \sigma_b^i \sigma_a^j \tau^k \sigma_b = \sigma_b^{i+1} \sigma_a^j \tau^{k-j}.$$

Thus we get $j = 0$. A similar computation shows that $i = 0$. Hence we showed $\text{Cent}(G) \subseteq \langle \tau \rangle$ and part (2) follows. \square

Remark 5.5. *There are projections:*

$$\pi_a : G \rightarrow G_a \quad \text{such that} \quad \pi_a(\sigma_a) = \sigma_a \quad \text{with kernel} \quad \langle \sigma_b, \tau \rangle.$$

$$\pi_b : G \rightarrow G_b \quad \text{such that} \quad \pi_b(\sigma_b) = \sigma_b \quad \text{with kernel} \quad \langle \sigma_a, \tau \rangle.$$

We are now ready to define a map

$$\varphi_{a,b} \in C^1(G) \quad \text{such that} \quad \partial(\varphi_{a,b}) = \chi_a \cup \chi_b \in C^2(G).$$

Let $\chi_a, \chi_b \in H^1(G)$ be the inflations to G , with respect to π_a, π_b , of the Kummer characters $\chi_a \in H^1(G_a), \chi_b \in H^1(G_b)$ corresponding to $a, b \in F^\times$.

Remark 5.6. *Note that by the above we have:*

$$\text{Ker}(\chi_a) = \langle \sigma_b, \tau \rangle \quad ; \quad \text{Ker}(\chi_b) = \langle \sigma_a, \tau \rangle.$$

Lemma 5.7. *Consider χ_a, χ_b as elements of $C^1(G)$.*

Let $g_1 = \sigma_b^i \sigma_a^j \tau^k, g_2 = \sigma_b^r \sigma_a^s \tau^t \in G$. We have:

$$\chi_a \cup \chi_b(g_1, g_2) = jr \quad \text{in } C^2(G).$$

Proof. We compute:

$$\begin{aligned} \chi_a \cup \chi_b(g_1, g_2) &= \chi_a(g_1) \chi_b(g_2) = \chi_a(\pi_a(g_1)) \chi_b(\pi_b(g_2)) = \\ &= \chi_a(\sigma_a^j) \chi_b(\sigma_b^r) = jr. \end{aligned}$$

\square

Define $\varphi_{a,b} \in C^1(G)$ via

$$\varphi_{a,b}(\sigma_b^i \sigma_a^j \tau^k) = k.$$

Proposition 5.8. *$\partial(\varphi_{a,b}) = \chi_a \cup \chi_b$ in $C^2(G)$, that is $\chi_a \cup \chi_b = 0$ in $H^2(G)$.*

Proof. Let $g_1, g_2 \in G$ be as above. We compute:

$$\begin{aligned}
\partial(\varphi_{a,b})(g_1, g_2) &= \partial(\varphi_{a,b})(\sigma_b^i \sigma_a^j \tau^k, \sigma_b^r \sigma_a^s \tau^t) \\
&= \varphi_{a,b}(\sigma_b^i \sigma_a^j \tau^k) + \varphi_{a,b}(\sigma_b^r \sigma_a^s \tau^t) - \varphi_{a,b}(\sigma_b^i \sigma_a^j \tau^k \sigma_b^r \sigma_a^s \tau^t) \\
&= k + t - \varphi_{a,b}(\sigma_b^{i+r} \sigma_a^{j+s} \tau^{k+t-rj}) = rj \\
&= \chi_a \cup \chi_b(g_1, g_2).
\end{aligned}$$

Thus $\partial(\varphi_{a,b}) = \chi_a \cup \chi_b$ in $C^2(G)$. \square

Corollary 5.9. *Let $\chi_w \in H^1(\text{Ker}(\chi_b))$ be the Kummer character corresponding to $w \in F_b$, that is*

$$\chi_w(\sigma_a) = 0 \quad ; \quad \chi_w(\tau) = 1$$

and consider it also as an element of $C^1(\text{Ker}(\chi_b))$. We have that

$$\text{res}_{\text{Ker}(\chi_b)}(\varphi_{a,b}) = \chi_w \in H^1(\text{Ker}(\chi_b)).$$

Proof. Let $g = \sigma_a^i \tau^j \in \text{Ker}(\chi_b)$, we compute:

$$\begin{aligned}
\text{res}_{\text{Ker}(\chi_b)}(\varphi_{a,b})(\sigma_a^i \tau^j) &= \varphi_{a,b}(\sigma_b^0 \sigma_a^i \tau^j) \\
&= j = \chi_w(\sigma_a^i \tau^j).
\end{aligned}$$

Thus $\text{res}_{\text{Ker}(\chi_b)}(\varphi_{a,b}) = \chi_w \in H^1(\text{Ker}(\chi_b))$. \square

6. PROOF OF THE MAIN THEOREM

In this section we will prove Theorem 6.6 stating that for any prime p , G_F has the Vanishing triple product property, using the previous sections.

Recall that we are given $\chi_a, \chi_b, \chi_c \in H^1(G_F)$ such that

$$\chi_a \cup \chi_b = \chi_b \cup \chi_c = 0.$$

Then, there exist $\varphi_{a,b}, \varphi_{b,c} \in C^1(G_F)$ such that

$$\partial(\varphi_{a,b}) = \chi_a \cup \chi_b \quad \text{and} \quad \partial(\varphi_{b,c}) = \chi_b \cup \chi_c$$

in $C^2(G_F)$. Hence,

$$\langle \chi_a, \chi_b, \chi_c \rangle_\varphi = \chi_a \cup \varphi_{b,c} + \varphi_{a,b} \cup \chi_c$$

is in $H^2(G_F)$, and

$$\langle \chi_a, \chi_b, \chi_c \rangle = \langle \chi_a, \chi_b, \chi_c \rangle_\varphi + \chi_a \cup H^1(G_F) + H^1(G_F) \cup \chi_c.$$

In order to use section 3, we need to assume χ_a, χ_c are linearly independent, and in order to use section 5 for $\chi_b \cup \chi_c = 0$ to find $\varphi_{b,c}$ we want to assume the last two slots (χ_b, χ_c) are linearly independent.

Proposition 6.1. *It is enough to prove Theorem 6.6 for the case: χ_a, χ_c are linearly independent, and χ_b, χ_c are linearly independent.*

Proof. Assume we know Theorem 6.6 for this case and consider the other cases.

Case 1. χ_a, χ_c are linearly dependent.

Without loss of generality we may assume $\chi_a = t\chi_c$ for some $t \in \mathbb{F}_p$.

It follows that $\text{Ker}(\chi_c) \subseteq \text{Ker}(\chi_a)$. Now given a defining system

$\varphi = \{\varphi_{a,b}, \varphi_{b,c}\}$, we compute,

$$\text{res}_{\text{Ker}(\chi_c)}(\langle \chi_a, \chi_b, \chi_c \rangle_\varphi) = \text{res}_{\text{Ker}(\chi_c)}(\chi_a) \cup \text{res}_{\text{Ker}(\chi_c)}(\varphi_{b,c}) + \text{res}_{\text{Ker}(\chi_c)}(\varphi_{a,b}) \cup \text{res}_{\text{Ker}(\chi_c)}(\chi_c) = 0.$$

By [16, Corollary 24.37], we get that $\langle \chi_a, \chi_b, \chi_c \rangle_\varphi \in \chi_c \cup H^1(G_F)$, and $0 \in \langle \chi_a, \chi_b, \chi_c \rangle$.

Case 2. χ_b, χ_c are linearly dependent.

First note that by the first case we may assume χ_a, χ_c are linearly independent. This implies that either $\chi_b = 0$ or χ_a, χ_b are linearly independent. If $\chi_b = 0$ we see that $\chi_a \cup \chi_b = 0; \chi_b \cup \chi_c = 0$ in $C^2(G_F)$, so we can choose $\varphi_{a,b} = \varphi_{b,c} = 0$ in $C^1(G_F)$ as a defining system and we get $0 \in \langle \chi_a, \chi_b, \chi_c \rangle$.

The remaining case is χ_a, χ_b are linearly independent. Note that by assumption $\langle \chi_a, \chi_b, \chi_c \rangle$ is defined, which implies that $\langle \chi_c, \chi_b, \chi_a \rangle$ is defined. Moreover, by the assumption of the Theorem we have $0 \in \langle \chi_c, \chi_b, \chi_a \rangle$. Let $\varphi = \{\varphi_{c,b}, \varphi_{b,a}\}$ be a defining system for $\langle \chi_c, \chi_b, \chi_a \rangle$ such that $\langle \chi_c, \chi_b, \chi_a \rangle_\varphi = 0$. Consider the Galois group G_{ab} , of $F[a^{\frac{1}{p}}, b^{\frac{1}{p}}]$ which by assumption is $\langle \sigma_a, \sigma_b \rangle \cong \mathbb{Z} \times \mathbb{Z}$ with the obvious action. Define two functions, $\psi_1, \psi_2 \in C^1(G_{ab})$ by,

$$\psi_1(\sigma_a^i \sigma_b^j) = -ij, \quad \psi_2(\sigma_a^i \sigma_b^j) = -j^2.$$

Note that we have $\text{res}_{\text{Ker}(\chi_a)}(\psi_1) = 0$. A direct computation shows that,

$$\partial(\psi_1) = \chi_a \cup \chi_b + \chi_b \cup \chi_a, \quad \partial(\psi_2) = 2\chi_b \cup \chi_b.$$

Now let $t \in \mathbb{F}_p$ be such that $\chi_c = t\chi_b$ and define $\varphi' = \{\varphi_{a,b}, \varphi_{b,c}\}$ by

$$\varphi_{a,b} = \psi_1 - \varphi_{b,a}, \quad \varphi_{b,c} = t\psi_2 - \varphi_{c,b}.$$

We compute:

$$\begin{aligned} \partial(\varphi_{a,b}) &= \partial(\psi_1) - \partial(\varphi_{b,a}) \\ &= \chi_a \cup \chi_b + \chi_b \cup \chi_a - \chi_b \cup \chi_a = \chi_a \cup \chi_b. \end{aligned}$$

$$\begin{aligned}
\partial(\varphi_{b,c}) &= \partial(t\psi_2) - \partial(\varphi_{c,b}) \\
&= 2t\chi_b \cup \chi_b - t\chi_b \cup \chi_b = t\chi_b \cup \chi_b = \chi_b \cup \chi_c.
\end{aligned}$$

Thus we see φ' is a defining system for $\langle \chi_a, \chi_b, \chi_c \rangle$. We compute,

$$\begin{aligned}
\langle \chi_a, \chi_b, \chi_c \rangle_{\varphi'} &= \chi_a \cup (t\psi_2 - \varphi_{c,b}) + (\psi_1 - \varphi_{b,a}) \cup \chi_c \\
&= t\chi_a \cup \psi_2 + \psi_1 \cup \chi_c - (\chi_a \cup \varphi_{c,b} + \varphi_{b,a} \cup \chi_c)
\end{aligned}$$

Now,

$$\begin{aligned}
\text{res}_{\text{Ker}(\chi_a)}(\langle \chi_a, \chi_b, \chi_c \rangle_{\varphi'}) &= t \cdot \text{res}_{\text{Ker}(\chi_a)}(\chi_a) \cup \text{res}_{\text{Ker}(\chi_a)}(\psi_2) + \\
&\quad \text{res}_{\text{Ker}(\chi_a)}(\psi_1) \cup \text{res}_{\text{Ker}(\chi_a)}(\chi_c) - \\
&\quad \text{res}_{\text{Ker}(\chi_a)}(\chi_a) \cup \text{res}_{\text{Ker}(\chi_a)}(\varphi_{c,b}) - \\
&\quad \text{res}_{\text{Ker}(\chi_a)}(\varphi_{b,a}) \cup \text{res}_{\text{Ker}(\chi_a)}(\chi_c) \\
&= \text{res}_{\text{Ker}(\chi_a)}(\varphi_{c,b}) \cup \text{res}_{\text{Ker}(\chi_a)}(\chi_a) + \\
&\quad \text{res}_{\text{Ker}(\chi_a)}(\chi_c) \cup \text{res}_{\text{Ker}(\chi_a)}(\varphi_{b,a}) \\
&= \text{res}_{\text{Ker}(\chi_a)}(\langle \chi_c, \chi_b, \chi_a \rangle_{\varphi}) = 0.
\end{aligned}$$

Thus by [16, Corollary 24.37], we get

$$\langle \chi_a, \chi_b, \chi_c \rangle_{\varphi'} \in \chi_a \cup H^1(G_F),$$

and $0 \in \langle \chi_a, \chi_b, \chi_c \rangle$. □

From now we assume χ_a, χ_b, χ_c satisfy χ_a, χ_c are linearly independent, and χ_b, χ_c are linearly independent. Applying the previous subsection, we see there is a defining system $\varphi = \{\varphi_{a,b}, \varphi_{b,c}\}$ such that

$$\text{res}_{\text{Ker}(\chi_c)}(\varphi_{b,c}) = \chi_{w_c}$$

where

$$w_c = \prod_{i=0}^{p-1} \sigma_c^i(v)^i \in F_c$$

for $v \in F_c$ such that $N_{F_c/F}(v) = b$.

Let A be the corresponding degree p^2 and exponent p abelian crossed product as in 4.2, we have:

Proposition 6.2. $[\text{res}_{F_c}(A)] = [(a, w_c)_{p, F_c}]$

Proof. By the naturality of the isomorphism Ψ from the introduction we have that, $[\text{res}_{F_c}(A)]$ corresponds to $\text{res}_{\text{Ker}_{\chi_c}}(\langle \chi_a, \chi_b, \chi_c \rangle_{\varphi}) =$

$\text{res}_{\text{Ker}_{\chi_c}}(\chi_a) \cup \text{res}_{\text{Ker}_{\chi_c}}(\varphi_{b,c}) = \text{res}_{\text{Ker}_{\chi_c}}(\chi_a) \cup \chi_{w_c}$, which in turn corresponds to $[(a, w_c)_{p, F_c}]$. \square

Note that since $\chi_a \cup \chi_b = 0$, we know there exist $u \in F_a$ such that $N_{F_a/F}(u) = b$, and we let

$$w_a = \prod_{i=0}^{p-1} \sigma_a^i(u)^i \in F_a.$$

Thus, we can construct the abelian crossed product from section 2,

$$A_{u,v} = \left(F_a F_c, \{ \sigma_a, \sigma_c \}, \{ w_c, w_a \}, \frac{v}{u} \right).$$

Recall this means $A_{u,v} = F_a F_c [z_a, z_c]$ with the following relations:

$$z_a k z_a^{-1} = \sigma_a(k), \quad z_c k z_c^{-1} = \sigma_c(k), \quad z_a^p = w_c, \quad z_c^p = w_a, \quad z_c z_a = \frac{v}{u} z_a z_c.$$

The next step is to show $[A]$ and $[A_{u,v}]$ only differ by the class of a symbol algebra $(c, s)_{F,p}$ for some $s \in F^\times$. To this end we first notice that:

Proposition 6.3. $[\text{res}_{F_c}(A_{u,v})] = [(a, w_c)_{p, F_c}]$.

Proof. By [16, Corollary 24.24] we have $[\text{res}_{F_c}(A_{u,v})] = [C_{A_{u,v}}(F_c)] = [F_a F_c [z_a]] = [(a, w_c)_{p, F_c}]$ as needed. \square

Corollary 6.4. *We have: $[A_{u,v}] = [A] \otimes [(c, s)_{p, F}]$ for some $s \in F^\times$.*

Proof. By [16, Corollary 24.37] it is enough to show that $[A_{u,v}] \otimes [A]^{-1}$ is split by F_c . But from the above we have $\text{res}_{F_c}([A_{u,v}] \otimes [A]^{-1}) = [(a, w_c)_{p, F_c}] \otimes [(a, w_c)_{p, F_c}]^{-1} = 1$. \square

Corollary 6.4 tells us that:

Proposition 6.5. *The abelian crossed product $A_{u,v}$ represents an element of $\langle \chi_a, \chi_b, \chi_c \rangle$.*

Proof. This is clear in light of Corollary 6.4 and Remark 2.5. \square

We can finally prove:

Theorem 6.6. *For any prime p and any field F containing a primitive p -th root of unity, G_F has the Vanishing triple Massey product property.*

Proof. By Remark 2.5, it is enough to show that there exist an element $\langle \chi_a, \chi_b, \chi_c \rangle_\varphi \in \langle \chi_a, \chi_b, \chi_c \rangle$ such that $\langle \chi_a, \chi_b, \chi_c \rangle_\varphi \in \chi_a \cup H^1(G_F) + H^1(G_F) \cup \chi_c$. Now by Proposition 6.5, we know the abelian crossed product $A_{u,v}$ represents some element $\langle \chi_a, \chi_b, \chi_c \rangle_\varphi \in \langle \chi_a, \chi_b, \chi_c \rangle$. Thus it is enough to show that $A_{u,v} \cong (a, s)_{F,p} \otimes (c, t)_{F,p}$ for some $s, t \in F^\times$, but this is exactly Theorem 3.6. \square

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